Lecture 8

- Review
  - static multipole expansion
  - radiation field
- Multipole expansion for radiation
  - Radiation region
  - Quasi-static region

Review of Lecture 7:

- We found the multipole expansion in the static case:

summed over

$$\varphi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{n!}\, Q_{i_1\ldots i_n}\frac{x_{i_1}\ldots x_{i_n}}{|\vec{x}|^{2n+1}}$$

$i_k = 1..3$

$Q_{i_1\ldots i_n}$ = 2-n-pole moment tensor of the charge density:

$$Q_{i_1\ldots i_n} = \int d^3x'\, \rho(x')\, T_{i_1\ldots i_n}(x')$$

$$T_{i_1\ldots i_n} = (2n-1)!!\, x_{i_1} x_{i_2}\ldots x_{i_n} - A_{i_1\ldots i_n}$$

$A_{i_1\ldots i_n}$ contains δ_{ij}'s so that to make $T_{i_1\ldots i_n}$ traceless with respect to all indices.

In the magnetic case we studied the leading ($n=1$) term:

$$\vec{A}^{(1)} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} + \mathcal{O}(|x|^{-3})$$

$$\vec{B}^{(1)} = \vec{\nabla} \times \vec{A}^{(1)} = \ldots = \frac{\mu_0}{4\pi} \cdot \frac{3\vec{n}(\vec{m} \cdot \vec{n}) - \vec{m}}{|\vec{x}|^3}$$

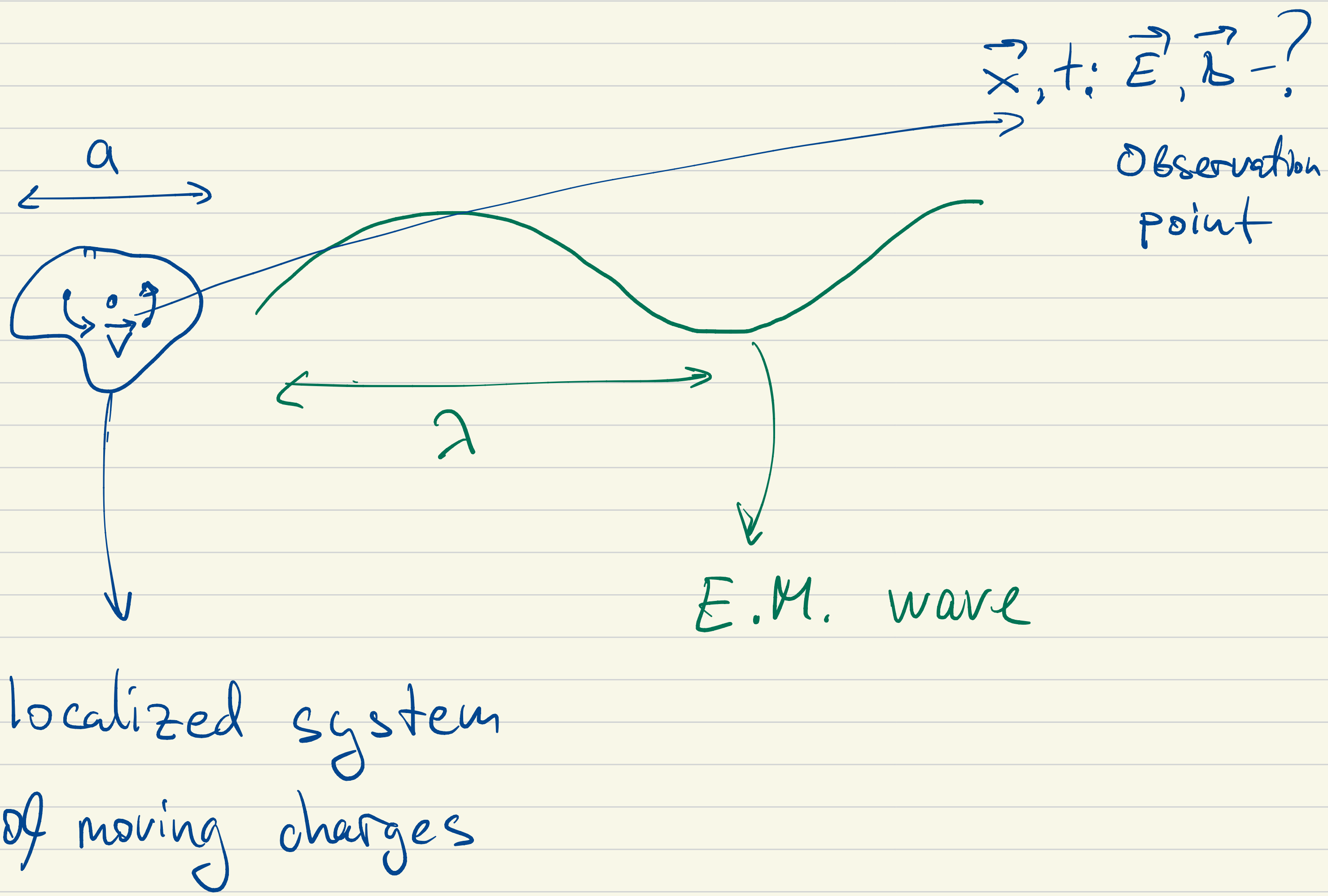
$$\vec{m} = \int_V dx' \vec{M}(x'),$$

where $\vec{M}(x) = \frac{1}{2} \vec{x}' \times \vec{J}(x')$

Note the similarity:

$$\vec{E}^{(1)} = \frac{1}{4\pi\varepsilon_0} \left(\frac{3\vec{n}(\vec{Q}^{(1)} \cdot \vec{n}) - \vec{Q}^{(1)}}{|\vec{x}|^3} \right).$$

Multipole expansion for E.M. radiation



There is a characteristic timescale $T = \frac{a}{v}$ in which charges move around the system. This gives the radiation of frequency $\omega \approx \frac{1}{T}$ and wavelength

$\lambda = \frac{c}{\omega} \simeq a\frac{c}{v}$. We will consider

$v << c \Rightarrow \boxed{\lambda >> a}$

It could also be that $v<<c$ but $\lambda << a$

[if there are many charges on complicated trajectories but we will not study this]

Also $|\vec{x}| >> a$ because we are interested in the fields far from the charges.

There are two cases then:

I. $a << \lambda << x$

II. $a << x << \lambda$

Let us study case I first (radiation region)

It will be enough to study the vector potential:

retarded G.F. (no radiation before charges existed)

$$\vec{A}(\vec{x},t) = \int dt'\, d^3x'\, \vec{J}(\vec{x}',t')\, G_r(\vec{x},t,\vec{x}',t') =$$

$$= \int dt'\, d\vec{x}'\, \vec{J}(\vec{x}',t')\, \frac{\mu_0}{4\pi} \cdot \frac{\delta\left(t - t' - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{\mu_0}{4\pi} \int d\vec{x}'\, \vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \frac{1}{|\vec{x}-\vec{x}'|}$$

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we would like to expand in

$$\frac{|\vec{x}'|}{|\vec{x}|} \sim \frac{a}{|\vec{x}|} \ll 1 \quad \text{and} \quad \frac{|\vec{x}'|}{\lambda} \sim \frac{a}{\lambda} \ll 1$$

Remember last lecture:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_n x'_{i_1} \ldots x'_{i_n} \frac{\partial}{\partial x'_{i_1}} \ldots \frac{\partial}{\partial x'_{i_n}} \frac{1}{|\vec{x}-\vec{x}'|}\Bigg|_{\vec{x}'=0} \ominus$$

$\frac{\partial}{\partial x'}$ produces one extra power of

$|\vec{x}|$ in the denominator.

$$\ominus \; \frac{1}{|\vec{x}|} + \frac{x'_i \cdot x_i}{|\vec{x}|^3} + \frac{1}{|\vec{x}|} \cdot O\left(\left(\frac{x'}{|\vec{x}|}\right)^2\right)$$

$$\frac{|\vec{x}-\vec{x}'|}{c} = \frac{|\vec{x}|}{c} - \underbrace{\frac{\vec{x}' \cdot \vec{x}}{c\,|\vec{x}|}}_{\text{keep this term, it is } O(1) \text{ in } \frac{1}{|\vec{x}|}\text{!}} + O\left(\frac{1}{|\vec{x}|}\right)$$

$$\vec{A}(\vec{x},t) =$$

$$= \frac{\mu_0}{4\pi} \cdot \frac{1}{|\vec{x}|} \cdot \int d^3\vec{x}' \, \vec{J}\left(\vec{x}', t - \frac{|\vec{x}|}{c} + \frac{\vec{x}' \cdot \vec{n}}{c}\right) \cdot \left[1 + O\left(\frac{a}{|\vec{x}|}\right)\right]$$

just the leading term in $\frac{1}{|\vec{x}-\vec{x}'|}$

$$= \frac{\mu_0}{4\pi|\vec{x}|} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3x' \left(\frac{\vec{x}' \cdot \vec{n}}{c}\right)^n \partial_t^n \vec{J}\left(x', t - \frac{x}{c}\right)$$

$$= \sum_{n=0}^{\infty} \vec{A}^{(n)} \longrightarrow$$ multipole expansion of the radiation.

$$\partial_t \vec{J} \sim \frac{1}{T} \sim \omega$$

$$\int \frac{(\vec{x}')^n \omega^n}{c^n} d\vec{x}' \simeq \left(\frac{a\omega}{c}\right)^n = \left(\frac{a}{\lambda}\right)^n$$

Let us find the magnetic field:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

in $\frac{a}{x}$ expansion

we focus on the leading, $n=0$, term:

$$\vec{A} = \frac{\mu_0}{4\pi |\vec{x}|} \cdot \int d^3\vec{x}' \, \vec{J}\left(x', t - \frac{|\vec{x}|}{c}\right)$$

$\vec{\nabla} \times \frac{1}{|\vec{x}|} \sim \frac{1}{|\vec{x}|^2}$, so the leading

term is when $\vec{\nabla}$ acts on $\vec{J}$:

$$\vec{\nabla} \times \vec{J}\left(x', t - \frac{|\vec{x}|}{c}\right) =$$

$$= \frac{\vec{n}}{c} \times \left(-\dot{\vec{J}}\left(x', t - \frac{|\vec{x}|}{c}\right)\right)$$

$$\vec{B} = \frac{-\mu_0}{4\pi |\vec{x}| c} \, \vec{n} \times \int d\vec{x}' \, \dot{\vec{J}}$$

We are at $x \gg a$ and $x \gg \lambda$, so the radiation is a plane wave, in direction $\vec{n}$ (away from the source) in which

$$\vec{B} = \frac{1}{c} \vec{n} \times \vec{E}$$

$$\vec{E} = -c \vec{n} \times \vec{B} \Rightarrow$$

$$\vec{E} = \frac{\mu_0}{4\pi |\vec{x}|} \vec{n} \times \vec{n} \times \int d\vec{x}' \, \dot{\vec{J}}\left(x', t - \frac{|\vec{x}|}{c}\right)$$

what is $\int d\vec{x}' \, \vec{J}(x', t)$?

$$\vec{J} = \rho \vec{v} = \sum_i e_i \dot{\vec{x}}_i$$

Now remember the definition of the dipole moment:

$$\vec{d} \equiv \vec{Q}^{(1)} = \int dx' \, \rho(x') \, \vec{x}' \quad \Rightarrow$$

$$\int d\vec{x}' \, \vec{\mathfrak{J}}(x', t) = \dot{\vec{d}}$$

$$\vec{B} = \frac{\mu_0}{4\pi c} \frac{1}{|\vec{x}|} \ddot{\vec{d}} \times \vec{n}$$

($\vec{d}$ at time $t - \frac{|\vec{x}|}{c}$)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{c}{\mu_0} |\vec{B}|^2 \vec{n} =$$

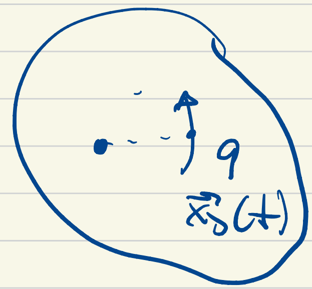
$$= \frac{1}{16\pi^2} \frac{\mu_0}{c} \left| \ddot{\vec{d}} \right|^2 \sin^2\theta \cdot \frac{\vec{n}}{x^2}$$

$$\frac{d\mathcal{E}}{dt} = \frac{1}{6\pi\varepsilon_0 c^3} \left| \ddot{\vec{d}} \right|^2$$

(after integrating over the angles)

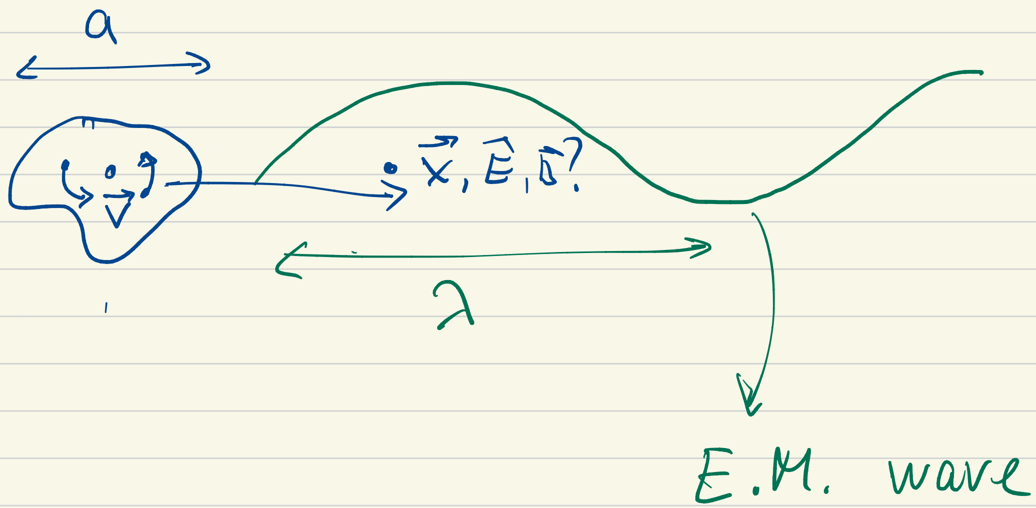
This is the Larmor formula we derived in the case of a single charge using Lienard-Wiechert potentials (in the non-relativistic, $v \ll c$ approx.):

$$\frac{d\mathcal{E}}{dt} = \frac{q^2 |\dot{\vec{\beta}}|^2}{6\pi \varepsilon_0 c} = \frac{q^2 \ddot{\vec{x}}_0^{\,2}}{6\pi \varepsilon_0 c^3}$$



Now let us consider briefly the other regime:

II. $a \ll x \ll \lambda$



In this case the subleading terms in $\left|\frac{a}{\vec{x}}\right|$ are more important than those in $\left|\frac{a}{\lambda}\right|$:

produced $\frac{a}{\lambda}$ expansion

$$A(\vec{x},t) =$$

(*)

$$= \frac{\mu_0}{4\pi} \int d\vec{x}' \, \vec{J}\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \frac{1}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{\mu_0}{4\pi} \left[\int d\vec{x}' \, \vec{J}(\vec{x}',t) \frac{1}{|\vec{x}-\vec{x}'|} \right] \left(1 + \mathcal{O}\left(\frac{a}{\lambda}\right) \right)$$

It is like the magnetostatic expansion but at each given moment of time:

$$\vec{A}(\vec{x},t) \simeq \sum_{n=1}^{\infty} \frac{a^n}{|\vec{x}|^{n+1}} \left(1 + O\left(\frac{a}{\lambda}\right)\right).$$

Same works for the scalar potential:

$$\varphi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\left[\int d_{x'} \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\right]\left(1+O\left(\frac{a}{\lambda}\right)\right)$$

$$= \sum_{n=0}^{\infty} \frac{Q_{i_1\dots i_n}(t)\, x_{i_1}\dots x_{i_n}}{|\vec{x}|^{2n+1}} \left(1+O\left(\frac{a}{\lambda}\right)\right) \simeq$$

$$\simeq \sum_{n=0}^{\infty} \frac{a^n}{|\vec{x}|^{2n+1}} \left(1+O\left(\frac{a}{\lambda}\right)\right)$$

Transition between the radiation region $x >> \lambda$ and quasi-static region $x << \lambda$ which happens at $x \sim \lambda$ is complicated even when both are larger than a.